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Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 21, NUMBER 5 (2018) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

FREQUENCY-DISTRIBUTED REPRESENTATION OF IRRATIONAL LINEAR SYSTEMS

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Abstract

The present work extends and generalizes the notion of frequency-distributed (FD) representation to a broad class of linear, stationary, continuous-time systems. On one hand, the proposed FD representation can be seen as a generalization of the diffusive representation, which is primarily utilized in the context of fractional order systems. Alternatively, it can also be seen as an extension to the Jordan canonical form, which is used as one of the main theoretical tools when analyzing finite-dimensional systems. Sufficient conditions under which FD representation can be achieved are derived. The proposed approach ensures real-valued state functions and output weights even when applied to oscillatory systems, and in a wast majority of cases manages to avoid utilization of generalized functions. Potential applications include simulation, representation theory and stability analysis, control synthesis, etc. All considerations have been illustrated by numerical examples.

MSC 2010: Primary 93B15; Secondary 93B10, 47A99, 47G10, 47G30

Key Words and Phrases: fractional order systems; frequency-distributed realization; infinite dimensional systems; generalized Jordan form; generalized Heaviside expansion

1. Introduction

Infinite-dimensional models are used throughout engineering and physics to describe various retarded, spatially-distributed, distributed-parameter

pp. 1396–1419 , DOI: 10.1515/fca-2018-0073 DE GRUYTER

and hereditary models. In the core of many such models are linear systems described by irrational transfer functions, i.e. those that cannot be represented as quotients of polynomials in the differentiation operator.

Detailed introduction into the theory of infinite-dimensional linear systems can be found in [7]. For an overview of transfer functions characteristic for various types of distributed-parameter systems we refer to [6]. In the past couple of decades fractional calculus has been recognized as a successful modeling tool in many scientific and engineering disciplines [30, 18]. Transfer functions corresponding to fractional order systems are also irrational, a property which is of significant interest in control [9]. Numerous models involving integro-differential operators of non-integer order have been proposed [39]. Control of such systems has become a vibrant research topic in the last decades, see e.g. [29, 17, 5, 34, 27]. More elaborate irrational models can be obtained by considering spatially-distributed fractional order systems [19, 4, 31, 8], distributed-order fractional models [14, 2, 12], complex-order models [3], and others.

Many common tasks are nontrivial when dealing with infinite-dimensional systems, including stability analysis and stabilization, simulation and realization. In light of these difficulties, the **diffusive representation** (DR) was introduced in 1990's with the aim of representing "non-standard" pseudo-differential operators by means of classical, albeit infinite dimensional, state-space models. It is most frequently used in the context of fractional order systems, for which it was originally developed [26]. However, it has been shown that DR can be effectively applied to a much broader class of linear operators of "diffusive" nature [25, 21].

According to [25, 21] a linear, time-invariant, causal system with impulse response (kernel, Green's function) $g(t)$ and transfer functions $G(s) = \mathcal{L}\{g(t)\}$ allows the diffusive representation *if* there exists $\gamma : [0, \infty) \to \mathbb{R}$ such that

$$g(t) = \int_0^\infty \gamma(\xi) e^{-\xi t} d\xi \; . \tag{1.1}$$

In this case, the process under consideration may be represented by the following **frequency-distributed (FD) state-space form**

$$\frac{\partial x(t,\xi)}{\partial t} = -\xi \, x(t,\xi) + u(t) \; , \tag{1.2}$$

$$y(t) = \int_0^\infty \gamma(\xi) x(t,\xi) d\xi \; , \tag{1.3}$$

where the output weight function γ is computed as

$$\gamma(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \mathrm{Im}\, G(-\xi - j\varepsilon) \; . \tag{1.4}$$

The frequency-distributed realization (1.2) may be seen as consisting of an infinite set of *completely decoupled* first order ordinary differential equations involving auxiliary variables $x(t,\xi)$. Thus, it can be regarded as a *proper state-space realization of an infinite-dimensional system*, with *"frequency-distributed"* state variables $x(t,\xi)$. Parameter ξ should be interpreted as the cut-off frequency (bandwidth) of the elementary sub-systems, or "states", in (1.2).

Several extensions and generalizations of the diffusive representation were considered [25]. In many cases DR is only possible provided that γ is interpreted as generalized function (a distribution, see [32]). "Diffusive Representation of the Second Kind", applicable to certain classes of oscillatory systems, was proposed in [10]. In this generalization, (1.2) and (1.3) should be replaced by

$$\frac{\partial x(t,\xi)}{\partial t} = -\left(\xi + j\omega\right)\, x(t,\xi) + u(t)\ , \tag{1.5}$$

$$y(t) = \mathrm{Re} \int_0^\infty \gamma(\xi) x(t,\xi) d\xi\ , \tag{1.6}$$

where j is the imaginary unit, $\omega > 0$ is fixed, and both the state function x and the output weight γ are allowed to be complex-valued. Diffusive representations of certain classes of infinite dimensional *discrete-time systems*, the so called "diffusive filters", were considered in [20, 10], while DR of fractional laplacian operator has been addressed in [23].

The diffusive representation was utilized very early in stability analysis of fractional order systems. Indeed, in [21] it was used as the cornerstone in the proof of the celebrated Matignon's Theorem. Stability of certain "generalized" forms of fractional systems, including discrete-time ones, was considered in [20]. A link towards application of Lyapunov and LaSalle techniques was further investigated in [22], while a Lyapunov-based approach to stability analysis of fractional differential equations, both linear and nonlinear ones, was considered in [37]. DR was also utilized to propose a natural solution to the problem of initialization of fractional order systems in [35]. For a related discussion regarding state variables and transients, we refer to [38, 36]. An approach to solving optimal control problems regarding FOS by means of DR was considered in [24]. The diffusive representation has also been extensively used in order to develop approximate, finite-dimensional models of fractional systems, see [26, 25].

The aim of the present work is to extend the notion of frequency-distributed representation to a broader class of systems. We investigate sufficient conditions under which FD representation can be achieved. We also demonstrate that, even in cases when representation (1.2), (1.3) is not applicable, a similar "decoupled" form involving more complex elementary dynamical systems associated with each value of cut-off frequency ξ can be obtained. Contrary to [10], the approach we are proposing ensures real-valued states and output weight functions even when applied to oscillatory systems. Contrary to [25], we manage to avoid utilization of generalized functions in a majority of cases. Indeed, frequency-distributed representation proposed in the present paper can be seen as a generalization of the Jordan canonical form [15, 1]. As a secondary result, the proposed approach also generalizes the notions of Heaviside's partial fractions decomposition [11, 33] and Prony's expansion [13].

The paper is organized in the following manner. This introductory section finishes with a brief account of the standard notation and terminology. A derivation of the (slightly generalized) diffusive representation is presented in Section 2, with a particular emphasis to the conditions under which DR is achievable. Extension to a class of oscillatory systems is proposed in Section 3. Section 4 proposes a modified FD representation in certain cases which do not satisfy conditions established in the previous sections. Some of these cases were also considered in [25], but the output weight had to be interpreted in the distributional sense. We managed to circumvent this by modifying the structure of the state-space equation (1.2). Unification of the various FD representation is considered in Section 5, which also discusses and emphasizes the fact that these representations are not unique. Connections to modal representation and the Jordan canonical form of finite-dimensional systems is discussed in Section 6. This Section also demonstrated the fact that distributional interpretation of the output weight is sometimes unavoidable. Final comments and possibilities for further work are given in the final Section 7.

Notation and terminology. Sets of real and complex numbers will be denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Imaginary unit will be denoted by j. For a complex number z, its conjugate number will be denoted by z^*. For a time-dependent signal (function) $g(t)$, its Laplace transform will be denoted by $G(s) = \mathcal{L}\{g(t)\}$. Similar notation will be used for the inverse Laplace transform, i.e. $g(t) = \mathcal{L}^{-1}\{G(s)\}$.

We recall that the transfer function of a linear, stationary system is defined as the ratio of Laplace transforms of its output and input, when all initial conditions are equal to zero. The inverse Laplace transform of the transfer function is referred to as the impulse response or kernel. The output of such a system is

$$y(t) = (g \star u)(t) = \int_0^t g(t-\tau)u(\tau)d\tau \ ,$$

where $\star$ is used to denote the operation of convolution. Less precise, but sometimes more convenient notation $g(t) \star u(t)$, will also be used.

2. The diffusive representation

In the present section we reiterate (a slightly generalized version of) frequency-distributed representation of systems having real-valued branching points [26, 25, 21, 20]. A particular attention will be paid to the conditions under which such representation is possible. Lessening of these constraints in the sequel will enable us to derive more general FD representations.

Assumption 1. The kernel $g(t)$ is real-valued. Consequently, $G(s)$ is symmetric w.r.t. to the real line, i.e. $G(s^*) = (G(s))^*$.

Assumption 2. $G(s)$ is analytic on $\mathbb{C} \setminus (-\infty, a]$ for some $a \in \mathbb{R}$.

Assumption 3. $\lim\limits_{s\to 0} sG(s+a) = 0$.

Assumption 4. $\lim\limits_{|s|\to\infty} G(s) = 0$.

Assumption 5. For each $\xi \in (-a, \infty)$ there exist well-defined limits G^+ and G^-

$$G^{\pm}(\xi) = \lim_{\varepsilon\to 0^+} G(-\xi \pm j\varepsilon) ,$$

and an absolutely integrable function $\overline{G}_\xi : \mathbb{R}^+ \to \mathbb{R}$ such that for almost all $t > 0$, $|G^{\pm}(\xi)e^{-\xi t}| < \overline{G}_\xi(t)$.

Theorem 2.1. *Under Assumptions 1–5, the impulse response (kernel) of the system described by transfer function $G(s)$ can be computed as*

$$g(t) = \int_{-a}^{\infty} \gamma(\xi) e^{-\xi t} d\xi \ , \tag{2.1}$$

yielding the diffusive representation (1.2), (1.4) *and*

$$y(t) = \int_{-a}^{\infty} \gamma(\xi) x(t,\xi) d\xi \ . \tag{2.2}$$

P r o o f. The kernel g, i.e. the inverse Laplace transform of G, can always be computed by means of the Fourier-Mellin inversion formula [33]

$$g(t) = \frac{1}{2\pi j} \int_{\ell - j\infty}^{\ell + j\infty} G(s) e^{st} d\, s \quad (\ell > a) \ . \tag{2.3}$$

Due to Assumptions 2 and 4, the Bromwich contour of (2.3) can be deformed as shown in Figure 1. In particular, this can be verified by slight modification of Jordan's Lemma [32], as has been done in [33]*. Modification of the contour gives rise to the following expression

$$g(t) = \frac{1}{2\pi j} \lim_{\varepsilon \to 0} \left(\int_{AB} + \int_{BC} + \int_{CD} \right) , \tag{2.4}$$

where integrands, which are the same as in (2.3), have been suppressed. It is not hard to show that $\int_{BC}$ vanishes. By setting $s = a + \varepsilon e^{j\varphi}$, one immediately finds that

$$\begin{aligned} \left| \int_{BC} \right| &= \left| \int_{-\pi/2}^{\pi/2} G(a + \varepsilon e^{j\varphi}) e^{(a+\varepsilon e^{j\varphi})t} \varepsilon e^{j\varphi} j d\varphi \right| \\ &\leq \int_{-\pi/2}^{\pi/2} \left| \varepsilon G(a + \varepsilon e^{j\varphi}) \right| e^{at} d\varphi \, . \end{aligned}$$

Due to Assumption 3, the inner absolute value will vanish for any φ as ε diminishes to zero, thus the entire integral vanishes also. Evaluating the remaining terms,

$$\begin{aligned} \int_{AB} + \int_{CD} &= - \int_{\infty}^{-a} G(-\xi - j\varepsilon) e^{-\xi t} d\xi - \int_{-a}^{\infty} G(-\xi + j\varepsilon) e^{-\xi t} d\xi \\ &= \int_{-a}^{\infty} \left[G(-\xi - j\varepsilon) - G(-\xi + j\varepsilon) \right] e^{-\xi t} d\xi \, . \end{aligned}$$

By Assumption 1, $G(-\xi - j\varepsilon) - G(-\xi + j\varepsilon) = 2j \mathrm{Im}\, \{G(-\xi - j\varepsilon)\}$, and

$$g(t) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{-a}^{\infty} \mathrm{Im}\, \{G(-\xi - j\varepsilon)\} e^{-\xi t} d\xi \, . \tag{2.5}$$

Finally, the limit and integration can interchange as a result of Assumption 5 and the celebrated Dominated Convergence Theorem [28], which immediately gives rise to (1.4).

Having in mind that the response of a linear, time-invariant system can be obtained by convolving the input with its impulse response, one obtains

$$y(t) = \int_0^{\infty} g(t - \tau) u(\tau) d\tau = \int_0^{\infty} \left(\int_{-a}^{\infty} \gamma(\xi) e^{-\xi(t-\tau)} d\xi \right) u(\tau) d\tau \, .$$

Finally, (1.3) is obtained by changing the order of integration, which is allowed by the [28], and introducing the *"internal"* variables

$$x(t, \xi) = \int_0^t e^{-\xi(t-\tau)} u(\tau) d\tau \, .$$

* Note that the assumption $|F(s)| \leq \frac{M}{|s|^p}$ for some $M > 0$, $p > 0$ which is used in [33] is not actually required. The proof presented in [33] remains valid if Assumption 4 (used also by the Jordan lemma itself) is introduced instead.

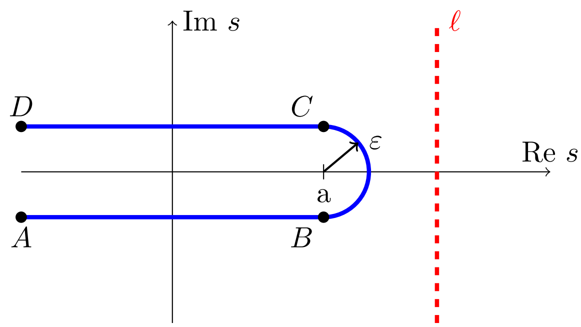

FIGURE 1. Integration contour used to derive generalized diffusive representation. The circular part of the contour with radius tending to infinity is omitted for clarity.

It can be verified in a fairly straightforward manner that the internal variables satisfy state equations (1.2), which concludes the proof. □

NOTE 1. Expression (2.1) can be seen as a generalization to Prony's expansion [13]. Its Laplace transform

$$G(s) = \int_{-a}^{\infty} \gamma(\xi) \frac{1}{s+\xi} d\xi \ ,$$

immediately generalizes Heaviside's partial fraction expansion.

3. Extension to a class of oscillatory systems

In the present section, the frequency-distributed representation will be extended to a class of systems having complex-conjugate branching points.

Assumption 2A. There exists $\omega \in \mathbb{R}$ such that all singularities of $G(s)$ belong to the union of two "horizontal" lines

$$(-\infty + j\omega, a + j\omega] \cup (-\infty - j\omega, a - j\omega]$$

for some $a \in \mathbb{R}$.

Assumption 3A. $\lim_{s \to 0} sG(s + a \pm j\omega) = 0$.

Assumption 5A. For each $\xi \in (-a, \infty)$ there exist well-defined limits G^+ and G^-

$$G^{\pm}(\xi) = \lim_{\varepsilon \to 0^+} G(-\xi + j(\omega \pm \varepsilon)) \ ,$$

and an absolutely integrable function $\overline{G}_\xi : \mathbb{R}^+ \to \mathbb{R}$ such that for almost all $t > 0$, $|G^{\pm}(\xi) e^{-\xi t}| < \overline{G}_\xi(t)$.

THEOREM 3.1. *Under assumptions 1, 2A, 3A, 4 and 5A, the impulse response (kernel) of the system described by transfer function $G(s)$ can be computed as*

$$g(t) = \int_{-a}^{\infty} \left[\gamma_c(\xi)\cos(\omega t) - \gamma_s(\xi)\sin(\omega t)\right] e^{-\xi t} d\xi\,, \tag{3.1}$$

where

$$\gamma_c(\xi) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Im}\,\{G(-\xi - j(\omega+\varepsilon)) - G(-\xi - j(\omega-\varepsilon))\}\,, \tag{3.2}$$

$$\gamma_s(\xi) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Re}\,\{G(-\xi - j(\omega+\varepsilon)) - G(-\xi - j(\omega-\varepsilon))\}\,; \tag{3.3}$$

yielding frequency-distributed representation

$$\frac{\partial x_c(t,\xi)}{\partial t} = -\xi x_c(t,\xi) - \omega x_s(t,\xi) + u(t) \tag{3.4}$$

$$\frac{\partial x_s(t,\xi)}{\partial t} = \omega x_c(t,\xi) - \xi x_s(t,\xi) \tag{3.5}$$

$$y(t) = \int_{-a}^{\infty} \left[\gamma_c(\xi) x_c(t,\xi) - \gamma_s(\xi) x_s(t,\xi)\right] d\xi\,. \tag{3.6}$$

P r o o f. The proof will be conducted in much the same way as the proof of Theorem 2.1, by deforming the Bromwich contour used for evaluating the impulse response g. In this case, however, the integration contour should be one depicted in Fig. 2. By means of the assumptions, the integral vanishes along the "small" semi-circles of radius ε, as well as along the "big" semicircle of radius R. Therefore

$$g(t) = \frac{1}{2\pi j} \lim_{\varepsilon\to 0} \left(\int_{D'C'} + \int_{B'A'} + \int_{AB} + \int_{CD} \right)\,.$$

By virtue of Assumption 5A, the Dominated Convergence Theorem holds, and the order of limit and integration can interchange. After some rearrangements, due to the continuity of the exponential function, one obtains

$$g(t) = \frac{1}{2\pi j} \int_{-a}^{\infty} \lim_{\varepsilon\to 0} \Big\{ \left[G(-\xi - j\omega - j\varepsilon) - G(-\xi - j\omega + j\varepsilon)\right] e^{(-\xi - j\omega)t} \\ - \left[G(-\xi + j\omega + j\varepsilon) - G(-\xi + j\omega - j\varepsilon)\right] e^{(-\xi + j\omega)t} \Big\}\, d\xi\,.$$

The first and the second term are, obviously, complex conjugate to each other. Therefore, the subtraction will cancel their real parts, and only double imaginary part will remain,

$$g(t) = \frac{1}{\pi} \int_{-a}^{\infty} \lim_{\varepsilon\to 0} \mathrm{Im}\,\left\{ \left[G(-\xi - j\omega - j\varepsilon) - G(-\xi - j\omega + j\varepsilon)\right] e^{(-\xi - j\omega)t} \right\} d\xi\,.$$

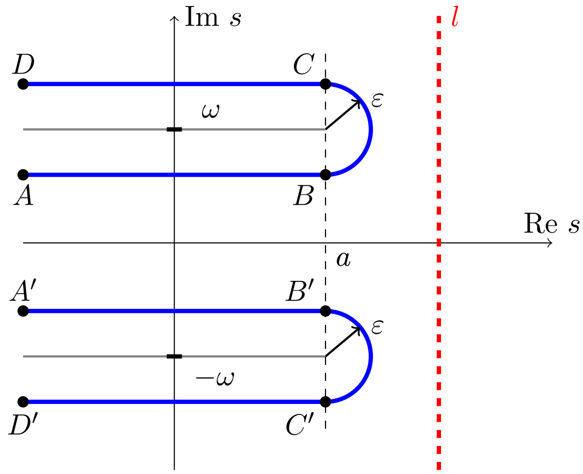

FIGURE 2. Integration contour used to derive diffusive representation in case when $G(s)$ has branch-cuts along horizontal lines with non-zero imaginary parts.

Expression (3.1) is obtained by introducing (3.2) and (3.3). By further defining

$$x_c(t,\xi) = e^{-\xi t}\cos(\omega t) \star u(t) \ , \tag{3.7}$$

$$x_s(t,\xi) = e^{-\xi t}\sin(\omega t) \star u(t) \ , \tag{3.8}$$

the frequency-distributed representation (3.4), (3.5), (3.6) is finally obtained. It is straightforward to check that x_c and x_s satisfy state equations (3.4) and (3.5). □

NOTE 2. Expressions (3.4) and (3.5) should be compared with (1.5) of the "Diffusive representation of the second kind" of Dauphin et al [10]. Similar comparison should be made between (1.3) and (1.6). While the representation presented in [10] preserves purely diagonal structure, the price to pay is that both states and output weight must be allowed to be complex-valued. In contrast, the representation proposed in the present paper is "block-diagonal", but purely real.

EXAMPLE 3.1. Consider operator described by transfer function $G(s) = \frac{1}{\sqrt{s^2+1}}$, with branching points $s = \pm j$. The kernel is the well-known Bessel function of the first kind $J_0(t)$ [16]. By direct computation one verifies that

$$\lim_{\varepsilon\to 0} G(-\xi - j(1\pm\varepsilon)) = \frac{\pm j}{\sqrt{\xi}\sqrt{-\xi-2j}} \ ,$$

and consequently

$$\lim_{\varepsilon\to 0}[G(-\xi-j(1+\varepsilon))-G(-\xi-j(1-\varepsilon))]=\frac{2j}{\sqrt{\xi}\sqrt{-\xi-2j}}$$
$$=\frac{-2}{\sqrt{\xi}\sqrt[4]{\xi^2+4}}e^{-\frac{j}{2}\arctan\frac{2}{\xi}}\ .$$

All assumptions of Theorem 3.1 are satisfied. In particular, the majorizing function of Assumption 5A can be selected as

$$\overline{G}(\xi,t)=\frac{C}{\sqrt{\xi}\sqrt[4]{\xi^2+4}}e^{-\xi t}\ ,$$

with $C>1$ sufficiently large. Expressions (3.2) and (3.3) reduce to

$$\gamma_c(\xi)=\frac{-2}{\pi\sqrt{\xi}\sqrt[4]{\xi^2+4}}\sin(\frac{1}{2}\arctan\frac{2}{\xi})\ ,$$
$$\gamma_s(\xi)=\frac{2}{\pi\sqrt{\xi}\sqrt[4]{\xi^2+4}}\cos(\frac{1}{2}\arctan\frac{2}{\xi})\ .$$

The corresponding integral representation of the Bessel function is obtained from (3.1) as

$$J_0(t)=J_{0,c}(t)-J_{0,s}(t)\ , \tag{3.9}$$

with

$$J_{0,c}(t)=\int_0^\infty\gamma_c(\xi)\cos(t)e^{-\xi t}d\xi\ ,$$
$$J_{0,s}(t)=\int_0^\infty\gamma_s(\xi)\sin(t)e^{-\xi t}d\xi\ .$$

The computation is illustrated by Fig. 3.

4. **Further generalizations**

The major obstacle in application of frequency-distributed representation in practice comes from Assumptions 3/3A and 5/5A. However, it often happens that although a specific transfer function does not satisfy these assumptions, its anti-derivative of a sufficiently high order does. The following claims extend the frequency-distributed representation to such cases.

THEOREM 4.1. *Let $G_0(s)$ be a transfer function satisfying all assumptions of Theorem 2.1. Let G be n-th anti-derivative of G_0 w.r.t. s for some $n\in\mathbb{N}$. The impulse response (kernel) of G can be computed as*

$$g(t)=(-t)^n\int_0^\infty\gamma_0(\xi)e^{-\xi t}d\xi\,, \tag{4.1}$$

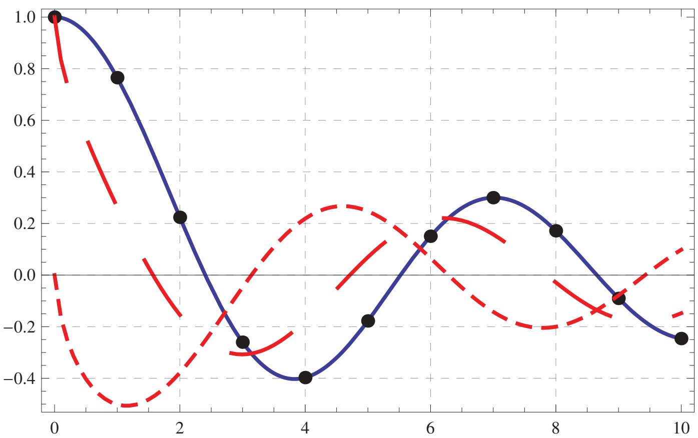

FIGURE 3. Evaluation of the Bessel function $J_0(t)$ (shown by thick line) by means of (3.9) (shown by thick circles). Long-dashed and short-dashed lines represent $J_{0,c}(t)$ and $J_{0,s}(t)$, respectively.

yielding frequency-distributed representation of the form

$$\frac{\partial x_1(t,\xi)}{\partial t} = -\xi x_1(t,\xi) + u(t) \,, \tag{4.2}$$

$$\frac{\partial x_i(t,\xi)}{\partial t} = -\xi x_i(t,\xi) + x_{i-1}(t,\xi) \,, \quad i \in \{2,\ldots,n+1\} \,, \tag{4.3}$$

$$y(t) = (-1)^n \, n! \int_0^\infty \gamma_0(\xi) x_{n+1}(t,\xi) d\xi \,, \tag{4.4}$$

where

$$\gamma_0(\xi) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \mathrm{Im}\,\{G_0(-\xi - j\,\varepsilon)\} \tag{4.5}$$

P r o o f. By assumption $G(s) = \left(\frac{d}{ds}\right)^n G_0(s)$. Differentiation in the complex domain is equivalent to multiplication by $-t$ in the time-domain, i.e. $g(t) = (-t)^n g_0(t)$ [33]. Since g_0 allows a diffusive representation,

$$\begin{aligned} g(t) = (-t)^n g_0(t) &= (-t)^n \int_0^\infty \gamma_0(\xi) e^{-\xi t} d\xi \\ &= (-1)^n \int_0^\infty \gamma_0(\xi) \left(t^n e^{-\xi t}\right) d\xi \end{aligned} \tag{4.6}$$

with γ_0 defined by (4.5). It is not hard to see that

$$t^n e^{-\xi t} = n! \, \underbrace{e^{-\xi t} \star e^{-\xi t} \star \ldots \star e^{-\xi t}}_{n+1 \text{ times}} \,. \tag{4.7}$$

Indeed, the Laplace transform of both sides in the above expression is $\frac{n!}{(s+\xi)^{n+1}}$. The response of G to input u can therefore be computed as

$$y(t) = (-1)^n \int_0^\infty \gamma_0(\xi) \left[\left(t^n e^{-\xi t} \right) \star u(t) \right] d\xi$$

$$= (-1)^n \, n! \int_0^\infty \gamma_0(\xi) \left[\underbrace{e^{-\xi t} \star e^{-\xi t} \star \ldots \star e^{-\xi t}}_{n+1 \text{ times}} \star u(t) \right] d\xi \,. \tag{4.8}$$

By choosing the internal variables as

$$x_1(t,\xi) = e^{-\xi t} \star u(t) \,,$$
$$x_i(t,\xi) = e^{-\xi t} \star x_{i-1}(t,\xi) \,, \quad i \in \{2, \ldots, n+1\} \,,$$

it is straightforward to check that relations (4.2) and (4.3) hold. □

EXAMPLE 4.1. Consider fractional integral of order $\beta \in (1,2)$, $G(s) = s^{-\beta}$. This transfer function does not allow for direct application of Theorem 2.1, but its first anti-derivative $G_0(s) = \frac{s^{1-\beta}}{1-\beta}$ does, with

$$\gamma_0(\xi) = \frac{\sin((\beta-1)\pi)}{\pi(1-\beta)\xi^{\beta-1}} = -\frac{\sin(\beta\pi)}{\pi(1-\beta)\xi^{\beta-1}} \,.$$

The corresponding frequency-distributed representation in the sense of Theorem 4.1 would be

$$\frac{\partial x_1(t,\xi)}{\partial t} = -\xi x_1(t,\xi) + u \,,$$
$$\frac{\partial x_2(t,\xi)}{\partial t} = -\xi x_2(t,\xi) + x_1(t,\xi) \,,$$
$$y(t) = -\int_0^\infty \gamma_0(\xi) x_2(t,\xi) d\xi \,.$$

Expression (4.1) reduces to

$$\frac{t^{\beta-1}}{\Gamma(\beta)} = t \int_0^\infty \frac{\sin(\beta\pi)}{\pi(1-\beta)} \frac{e^{-\xi t}}{\xi^{\beta-1}} d\xi \,. \tag{4.9}$$

The computation is illustrated by Fig. 4.

THEOREM 4.2. *Let $G_0(s)$ be a transfer function satisfying all assumptions of Theorem 3.1. Let G be n-th anti-derivative of G_0 w.r.t. s for some $n \in \mathbb{N}$. The impulse response (kernel) of G can be computed as*

$$g(t) = (-t)^n \int_a^\infty \left[\gamma_{0,c}(\xi) e^{-\xi t} \cos(\omega t) - \gamma_{0,s}(\xi) e^{-\xi t} \sin(\omega t) \right] d\xi \,, \tag{4.10}$$

yielding frequency-distributed representation of the form

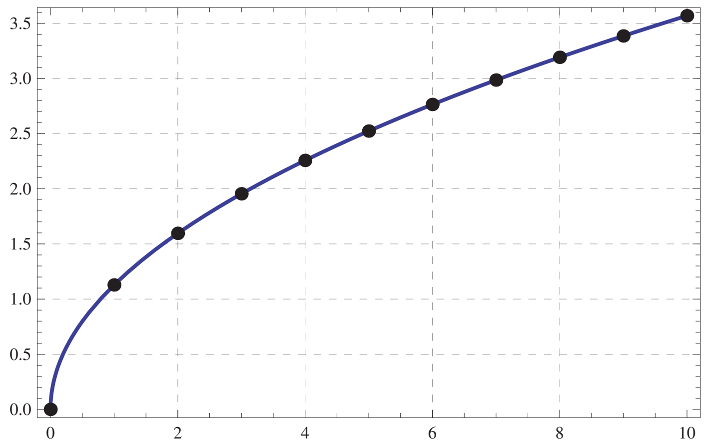

FIGURE 4. Evaluation of the power kernel $t^{\beta-1}/\Gamma(\beta)$ for $\beta = 1.5$ (shown by thick line) by means of (4.9) (shown by filled circles).

$$\frac{\partial x_{1,c}(t,\xi)}{\partial t} = -\xi x_{1,c}(t,\xi) - \omega x_{1,s}(t,\xi) + u(t) \tag{4.11}$$

$$\frac{\partial x_{1,s}(t,\xi)}{\partial t} = \omega x_{1,c}(t,\xi) - \xi x_{1,s}(t,\xi) \tag{4.12}$$

$$\frac{\partial x_{i,c}(t,\xi)}{\partial t} = -\xi x_{i,c}(t,\xi) - \omega x_{i,s}(t,\xi) + x_{i-1,c}(t,\xi)\ , \quad i \in \{2,\dots,n+1\} \tag{4.13}$$

$$\frac{\partial x_{i,s}(t,\xi)}{\partial t} = \omega x_{i,c}(t,\xi) - \xi x_{i,s}(t,\xi) + x_{i-1,s}(t,\xi)\ , \quad i \in \{2,\dots,n+1\} \tag{4.14}$$

$$y(t) = (-1)^n\, n! \int_a^\infty \left[\gamma_{0,c}(\xi) x_{c,n+1}(t,\xi) - \gamma_{0,s}(\xi) x_{s,n+1}(t,\xi)\right] d\xi\ , \tag{4.15}$$

where

$$\gamma_{0,c}(\xi) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im}\left\{G_0(-\xi - j(\omega+\varepsilon)) - G_0(-\xi - j(\omega-\varepsilon))\right\}\ , \tag{4.16}$$

$$\gamma_{0,s}(\xi) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Re}\left\{G_0(-\xi - j(\omega+\varepsilon)) - G_0(-\xi - j(\omega-\varepsilon))\right\}\ . \tag{4.17}$$

P r o o f. By assumption $G(s) = \left(\frac{d}{ds}\right)^n G_0(s)$. Differentiation in the complex domain is equivalent to multiplication by $-t$ in the time-domain, i.e. $g(t) = (-t)^n g_0(t)$ [33]. Since g_0 allows a frequency-distributed representation,

$$g(t) = (-1)^n n! \int_a^\infty \left[\gamma_{0,c}(\xi)\frac{t^n}{n!}e^{-\xi t}\cos(\omega t) - \gamma_{0,s}(\xi)\frac{t^n}{n!}e^{-\xi t}\sin(\omega t)\right] \quad (4.18)$$

with $\gamma_{0,c}$ and $\gamma_{0,s}$ defined by (4.16) and (4.17), respectively. Let us define

$$x_{i,c}(t,\xi) = \frac{t^{i-1}}{(i-1)!}e^{-\xi t}\cos(\omega t) \star u(t) \ , \quad (4.19)$$

$$x_{i,s}(t,\xi) = \frac{t^{i-1}}{(i-1)!}e^{-\xi t}\sin(\omega t) \star u(t) \ , \quad (4.20)$$

for all $i \in \{1, \dots, n+1\}$. It can be easily verified that

$$\frac{d}{dt}f(t) \star h(t) = \dot{f}(t) \star h(t) + f(0) \star h(t) \ . \quad (4.21)$$

Consequently, for $i \in \{2, \dots, n+1\}$

$$\begin{aligned} \frac{\partial x_{i,c}(t,\xi)}{\partial t} &= \frac{t^{i-1}}{(i-1)!}e^{-\xi t}\cos(\omega t) \star u(t) \ , \\ &= \left[-\xi\frac{t^{i-1}}{(i-1)!}e^{-\xi t}\cos(\omega t) - \omega\frac{t^{i-1}}{(i-1)!}e^{-\xi t}\sin(\omega t) \right. \\ &\qquad \left. + \frac{t^{i-2}}{(i-2)!}e^{-\xi t}\cos(\omega t)\right] \star u(t) \\ &= -\xi x_{i,c}(t,\xi) - \omega x_{i,s}(t,\xi) + x_{i-1,c}(t,\xi) \ . \end{aligned}$$

Thus, expression (4.13) is derived. Expressions (4.11), (4.12) and (4.14) follow by similar arguments. The output equation (4.15) is obtained directly by inserting (4.19) and (4.20) into (4.18) and convolving with the input signal u. □

5. **Unification and alternative representations**

All frequency-distributed representations considered thus far can be cast in a unique framework. Indeed, the essence of this representation is to decompose the system under consideration $G(s)$ into an *infinite parallel connection of simple, finite dimensional sub-systems* $G(s,\xi)$, indexed by a continuous real parameter ξ. The overall output is then constructed by superposition of outputs of individual sub-systems, according to a particularly selected output weighting function. Different frequency-distributed representations, as considered in Theorems 2.1, 3.1, 4.1 and 4.2, are obtained due to variations in structure and complexity of individual sub-systems under different conditions. Indeed, all of them can be obtained by applying classical realization theory [11, 1, 15] to $G(s,\xi)$ for each fixed ξ.

In the present section, we propose a unified representation

$$y(t) = \int_{-\infty}^{\infty} \varrho(\xi) y(t,\xi) d\xi \ , \tag{5.1}$$

$$y(t,\xi) = g(t,\xi) \star u(t) \ , \tag{5.2}$$

where $g(t,\xi)$ is the impulse response of $G(s,\xi)$, i.e. $g(t,\xi) = \mathcal{L}^{-1}\{G(s,\xi)\}$, and $y(t,\xi)$ is its response to the common input signal $u(t)$. The output weighting function ϱ is closely related to the previously considered output weights γ, γ_s, γ_c, etc., but is not a direct substitute for any of them, as will be discussed in the sequel. The proposed unified representation also implies that

$$g(t) = \int_{-\infty}^{\infty} \varrho(\xi) g(t,\xi) d\xi \ , \tag{5.3}$$

$$G(s) = \int_{-\infty}^{\infty} \varrho(\xi) G(s,\xi) d\xi \ . \tag{5.4}$$

These expressions fully generalize expansions of Prony and Heaviside, respectively.

The diffusive representation (1.2), (1.3) established by Theorem 2.1 is recovered by setting $G(s,\xi) = \frac{1}{s+\xi}$ and $y(t,\xi) = x(t,\xi)$. With a slight abuse of notation, one may write

$$\varrho(\xi) = \gamma(\xi) h(\xi + a) \equiv \begin{cases} \gamma(\xi) & , \ \xi > -a \\ 0 & , \ \text{otherwise} \end{cases} ,$$

where h is Heaviside's unit step function [11]. In this simplest case, the individual sub-components are first order systems and the diffusive representation can be considered as a diagonal realization of an irrational transfer function. The diffusive representation is not unique. It would also be possible to consider $G(s,\xi) = \frac{\gamma(\xi)}{s+\xi}$, yielding $\varrho(\xi) = h(\xi + a)$, and

$$\frac{\partial x(t,\xi)}{\partial t} = -\xi x(t,\xi) + \gamma(\xi) u(t) \ .$$

Also, provided that γ is positive for all ξ, it is possible to choose $G(s,\xi) = \frac{\sqrt{\gamma(\xi)}}{s+\xi}$, with $\varrho(\xi) = \sqrt{\gamma(\xi)} h(\xi + a)$, and

$$\frac{\partial x(t,\xi)}{\partial t} = -\xi x(t,\xi) + \sqrt{\gamma(\xi)} u(t) \ .$$

Frequency-distributed representation of oscillatory systems (3.4), (3.5) and (3.6), considered in Theorem 3.1, is recovered by choosing

$$G(s,\xi) = \frac{\gamma_c(\xi)(s+\xi) + \gamma_s(\xi)\omega}{(s+\xi)^2 + \omega^2} \ ,$$

and $\varrho(\xi) = h(\xi + a)$, and selecting the state variables according to (3.7), (3.8). The corresponding elementary output is $y(t,\xi) = \gamma_c(\xi)x_c(t,\xi) + \gamma_s(\xi)x_s(t,\xi)$. This is, off course, not a unique way of choosing elementary state variables. By opting for the observable canonical form [11, 1, 15], for example, an alternative frequency distributed realization of a transfer function satisfying all assumptions of Theorem 3.1 would become

$$\begin{aligned}\frac{\partial x_1(t,\xi)}{\partial t} &= \qquad\qquad -2\xi x_1(t,\xi + x_2(t,\xi) + \gamma_c(\xi)u(t) \ , \\ \frac{\partial x_2(t,\xi)}{\partial t} &= -(\xi^2+\omega^2)x_1(t,\xi) + (\gamma_x(\xi)\xi + \gamma_s(\xi)\omega)\, u(t) \ ,\end{aligned}$$

with $y(t,\xi) = x_1(t,\xi)$ and $\varrho(\xi) = h(\xi + a)$. Obviously, other options are available.

Diffusive representation (4.2), (4.3), (4.4) considered in Theorem 4.1 is recovered by considering elementary transfer functions

$$G(s,\xi) = \frac{1}{(s+\xi)^{n+1}} \ ,$$

and setting $\varrho(\xi) = (-1)^n n! \gamma_0(\xi) h(\xi + a)$. Since the elementary system has repeated real poles, expressions (4.2)–(4.4) should be seen as a generalization of the Jordan canonical form to transfer functions of infinite dimension. The "signature structure" of eigenvalues on the main diagonal and ones on the super-diagonal is clearly preserved. Other realizations are possible, including observable or controllable canonical forms.

Similar considerations can be made for the case covered by Theorem 4.2. In this last case the elementary system has repeated complex-conjugate poles.

It is important to realize that the frequency-distributed realization we are just describing is applicable to systems having several types of different singularities. In such a case, even the structure of the elementary systems changes with ξ. This situation is illustrated by the following example.

Example 5.1. Consider a system described by $G(s) = \frac{1}{\sqrt{s^2+1}\sqrt{s+1}}$, having branching points at -1 and $\pm j$. Using elementary properties of the Laplace transform [33], we find that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = J_0(t) \star \frac{e^{-t}}{\sqrt{\pi t}} \ . \tag{5.5}$$

The FD representation can be derived by considering integration contour shown in Fig. 5. By repeating the considerations used in proofs of Theorems 2.1 and 3.1, it is easily seen that in the case under consideration the elementary transfer functions have the following form

$$G(s,\xi) = \begin{cases} G_1(s,\xi) & , \ \xi \in [0,1) \\ G_1(s,\xi) + G_2(s,\xi) & , \ \xi \geq 1 \end{cases} ,$$

where $G_1(s,\xi) = \frac{\gamma_{1,c}(\xi)(s+\xi)+\gamma_{1,s}(\xi)}{(s+\xi)^2+1}$ and $G_2(s,\xi) = \frac{\gamma_2(\xi)}{s+\xi}$. Weight function γ_2 is computed according to (1.4), yielding

$$\gamma_2(\xi) = \frac{\sin(\frac{\pi}{2})}{\pi\sqrt{\xi-1}\sqrt{\xi^2+1}} ,$$

while $\gamma_{1,c}$ and $\gamma_{1,s}$ are computed according to (3.2) and (3.3), respectively, as imaginary and real part of

$$\gamma_1(\xi) = \frac{2j}{\pi\sqrt{\xi}\sqrt{-\xi-2j}\sqrt{1-\xi-j}} .$$

The overall frequency-distributed representation is

$$\begin{aligned} \frac{\partial x_{1,c}(\xi,t)}{\partial t} &= -\xi x_{1,c}(\xi,t) - \omega x_{1,s}(\xi,t) + u(t) \\ \frac{\partial x_{1,s}(\xi,t)}{\partial t} &= \omega x_{1,s}(\xi,t) - \xi x_{1,c}(\xi,t) \\ \frac{\partial x_2(t,\xi)}{\partial t} &= -\xi\, x_2(t,\xi) + u(t) , \\ y_1(t) &= \int_0^\infty \left[\gamma_{1,c}(\xi)x_{1,c}(\xi,t) - \gamma_{1,s}(\xi)x_{1,s}(\xi)\right] d\xi . \\ y_2(t) &= \int_1^\infty \gamma_2(\xi)x_2(t,\xi)d\xi , \\ y(t) &= y_1(t) + y_2(t) . \end{aligned}$$

The alternative way of computing (5.5) is (as illustrated in Fig. 5)

$$g(t) = \int_0^\infty \left[\gamma_{c,1}(\xi)\cos(t) - \gamma_{s,1}(\xi)\sin(t)\right] e^{-\xi t} d\xi + \int_1^\infty \gamma_2(\xi) e^{-\xi t} d\xi . \quad (5.6)$$

6. **Singular Representations**

In the present section we show that in the cases when the transfer function under consideration has poles, the output weight functions must contains impulses, and have to be interpreted as a generalized function [32]. Particularly, when the system under consideration is finite-dimensional, i.e. when the corresponding function is rational, the frequency-distributed representation reduces to the classical state-space model in Jordan form [1, 15]. For simplicity, only systems having simple poles, real or complex, are

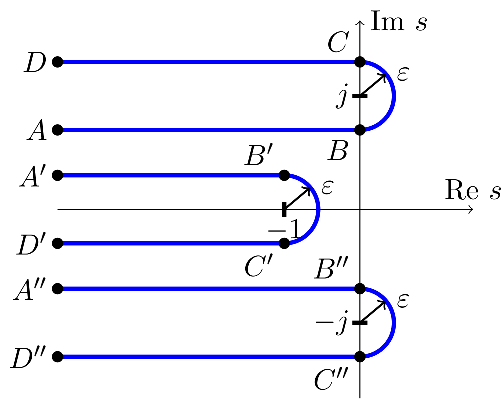

FIGURE 5. Integration contour used in Example 5.1.

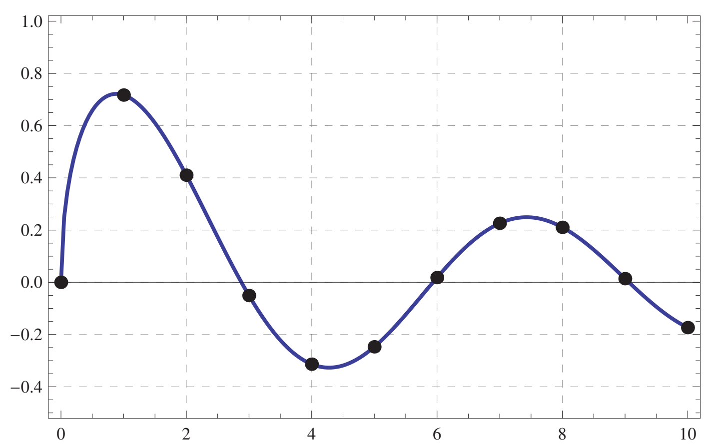

FIGURE 6. The comparison of the impulse response $g(t)$ considered in Example 5.1 computed by means of eq. (5.6) (filled dots) and eq. (5.5) (thick line).

considered in the present section. The results can be extended to systems with repeated poles in a straightforward manner.

6.1. **Systems with simple, real poles.** Consider a strictly proper transfer function $G(s)$ having simple real poles $p_k \in \mathbb{R}$, $k \in \{1, \ldots, n\}$, where n may be finite or infinite. Such a transfer function can always be written in the form

$$G(s) = \sum_{i=k}^{n} \frac{K_k}{s - p_k}\ , \tag{6.1}$$

where K_k are real coefficients obtained by the partial fractions expansion procedure. It is not hard to see that such a transfer function satisfies Assumptions 1 to 4 with $a = \max_k p_k$.

Since Assumption 5 is not satisfied, one is no longer allowed to apply the Dominated Convergence Theorem to (2.5). However, by inserting (6.1) into (2.5) instead,

$$\begin{aligned} g(t) &= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int_{-a}^{\infty} \mathrm{Im} \left\{ \sum_{k=1}^{n} \frac{K_k}{-\xi - j\varepsilon - p_k} \right\} e^{-\xi t} d\xi \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int_{-a}^{\infty} \sum_{k=1}^{n} \frac{K_k \varepsilon}{(\xi + p)^2 + \varepsilon^2} e^{-\xi t} d\xi \,. \end{aligned}$$

It is well-known result from the theory of generalized functions [32] that the so called Cauchy kernel $\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$ weakly converges to the Dirac's δ-distribution, therefore

$$g(t) = \int_{-a}^{\infty} \sum_{k=1}^{n} K_k \delta(\xi + p_k) e^{-\xi t} d\xi = \sum_{k=1}^{n} K_i e^{p_k t} \,.$$

The last expression implies that one can define

$$\gamma(\xi) = \sum_{k=1}^{n} K_k \delta(\xi + p_k) \tag{6.2}$$

and formally retain the same form of the diffusive representation (1.2)–(1.3). Clearly, the only relevant state variables are $x_k(t) = x(-p_k, t)$, with dynamics

$$\dot{x}_k(t) = p_k x_k(t) + u(t) \,. \tag{6.3}$$

In this particular case, the frequency-distributed representation reduces to the classical, diagonal state-space realization of rational transfer functions. The method is, however, completely valid even in the case of systems having infinitely many discrete poles, as illustrated by the following example.

Example 6.1. Consider transfer function

$$G(s) = \frac{\sinh(\beta\sqrt{s})}{\sinh(\sqrt{s})} \,,$$

with $\beta \in (0, 1)$, which has simple poles at $p_k = -(k\pi)^2$ for all $k \in \mathbb{N}$. Since all poles are simple, coefficients of the Heaviside's expansion can be computed as

$$K_k = \lim_{s\to -(k\pi)^2} (s + (k\pi)^2) G(s) = (-1)^{k-1} 2\pi k \sin(\beta\pi k) \,,$$

which implies

$$\frac{\sinh(\beta\sqrt{s})}{\sinh(\sqrt{s})} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2\pi k \sin(\beta\pi k)}{s + (k\pi)^2} \, .$$

The corresponding diffusive representation is actually an infinite-dimensional, diagonal state-space model

$$\begin{aligned} \dot{x}_k(t) &= -(k\pi)^2 x_k(t) + u(t) \, , \\ y(t) &= \sum_{k=1}^{\infty} (-1)^{k-1} 2\pi k \sin(\beta\pi k) x_k(t) \, . \end{aligned}$$

6.2. **Systems with complex-conjugate poles.** Consider a strictly proper transfer function $G(s)$ having complex-conjugate pole pairs $p_k \pm j\omega$, for some $\omega > 0$ and $p_k \in \mathbb{R}$, $k \in \{1, \ldots, n\}$, where n may be finite or infinite. The corresponding Heaviside's expansion is

$$G(s) = \sum_{k=1}^{n} \frac{\alpha_k + j\beta_k}{s - p_k - j\omega} + \frac{\alpha_k - j\beta_k}{s - p_k + j\omega} \, , \tag{6.4}$$

with $\alpha_k, \beta_k \in \mathbb{R}$. Such a transfer function satisfies Assumptions 1, 2A, 3A and 4 with $a = \max_k p_k$, but not 5A.

One can now re-trace all the steps of the proof of Theorem 3.1, and identify the point at which instead of applying the Dominated Convergence Theorem, weak convergence of the Cauchy kernel would be utilized. Instead of this, a more elementary, direct approach can be utilized. Eq. (6.4) can be re-written as

$$G(s) = \sum_{k=1}^{n} \frac{2\alpha_k (s - p_k) - 2\beta_k \omega}{(s - p_k)^2 + \omega^2} \, ,$$

implying impulse response in the form

$$g(t) = \sum_{k=1}^{n} e^{p_k t} \left[2\alpha_k \cos(\omega t) - 2\beta_k \sin(\omega t) \right] \, , \tag{6.5}$$

which is formally equivalent to (3.1) with

$$\begin{aligned} \gamma_c(\xi) &= \sum_{k=1}^{n} 2\alpha_k \delta(\xi + p_k) \, , \\ \gamma_s(\xi) &= \sum_{k=1}^{n} 2\beta_k \delta(\xi + p_k) \, . \end{aligned}$$

7. Conclusions

Frequency-distributed representation derived in the present paper extends the notion of the diffusive representation previously derived and utilized primarily in the context of fractional order systems. The proposed representation is applicable to a wide class of linear systems, including those with complex-conjugate pairs of singularities. Potential applications span from developing finite-dimensional approximations, both continuous-time and discrete-time ones, to stability analysis and control design. Closer investigation of these issues goes beyond the scope of the present work and will be pursued further in the future.

Acknowledgments

The authors kindly acknowledge the support of Ministry of Education, Science and Technological Development under grants TR32018 and TR33013 (MRR) and TR33020 (TBŠ).

The authors' work is also in frames of the working program on bilateral agreement between Serbian and Bulgarian Academies of Sciences, SASA – BAS, 2017-2019.

References

[1] P.J. Antsaklis, A.N. Michel, *A Linear Systems Primer.* Birkhäuser (2007).

[2] T.M. Atanacković, M. Budinčević, S. Pilipović, On a fractional distributed-order oscillator. *Journal of Physics A* **38**, No 30 (2005), 6703–6713; doi: 10.1088/0305-4470/38/30/006.

[3] T.M. Atanacković, S. Konjik, S. Pilipović, D. Zorica, Complex order fractional derivatives in viscoelasticity. *Mechanics of Time-Dependent Materials* **20**, No 2 (2016), 175–195; doi: 10.1007/s11043-016-9290-3.

[4] T.M. Atanacković, S. Pilipović, D. Zorica, Diffusion wave equation with two fractional derivatives of different order. *J. Phys. A Math. Theor.* **40** (2007), 5319–5333; doi:10.1088/1751-8113/40/20/006.

[5] R. Capponeto, S. Graziani, V. Tomasello, A. Pisano, Identification and fractional super-twisting robust control of IPMC actuators. *Fract. Calc. Appl. Anal.* **18**, No 6 (2015), 1358–1397; doi: 10.1515/fca-2015-0079; at https://www.degruyter.com/view/j/fca.

[6] R. Curtain, K. Morris, Transfer functions of distributed parameter systems: a tutorial. *Automatica*, **45**, No 5 (2009), 1101–1116; doi: 10.1016/j.automatica.2009.01.008.

[7] R.Z. Curtain, H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, New York, Inc. (1995).

[8] S. Cvetićanin, D. Zorica, M.R. Rapaić, Generalized time-fractional telegrapher's equation in transmission line modeling. *Nonlinear Dynamics* **88**, Issue 2 (2017), 1453–1472, doi: 10.1007/s11071-016-3322-z.

[9] D. Valerio, J.S. da Costa, *An Introduction to Fractional Control.* IET, London, UK (2013).

[10] G. Dauphin, D. Heleschewitz, D. Matignon, Extended diffusive representations and application to non-standard oscillators. In: *Proc. of Mathematical Theory on Network Systems (MTNS).* University of Perpignan, Perpignan, France (2000).

[11] R.C. Dorf, R.H. Bishop, *Modern Control Systems.* Pearson Prentice Hall (2011).

[12] R. Gorenflo, Y. Luchko, M. Stojanović, Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density. *Fract. Calc. Appl. Anal.* **16**, No 2 (2013), 297–316; doi: 10.2478/s13540-013-0019-6; at https://www.degruyter.com/view/j/fca.

[13] J.F. Hauer, C.J. Demeure, L.L. Scharf, Initial results in Prony analysis of power system response signals. *IEEE Trans. on Power Systems* **5**, No 1 (1990), 80–89; doi:10.1109/59.49090.

[14] Z. Jiao, Y.Q. Chen, I. Podlubny, *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives.* Springer (2012).

[15] T. Kailath, *Linear Systems.* Prentice-Hall (1980).

[16] G. Korenev, *Bessel Functions and Their Applications.* CRC Press (2002).

[17] P. Lanusse, J. Sabatier, PLC implementation of a CRONE controller. *Fract. Calc. Appl. Anal.* **14**, No 4 (2011), 505–522; doi: 10.2478/s13540-011-0031-7; at https://www.degruyter.com/view/j/fca.

[18] R.L. Magin, *Fractional Calculus in Bioengineering.* Begell House Publishers (2006).

[19] F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space–time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4**, No 2 (2001), 153–192.

[20] D. Matignon, Generalized fractional differential and difference equations: Stability properties and modeling issues. In: *Proc.of Mathematical Theory of Networks and Systems*, Padova, Italy, Il Poligrafo (1998), 503–506.

[21] D. Matignon, Stability properties for generalized fractional differential systems. In: *Proc. of ESAIM Fractional Differential Systems: Models, Methods and Applications*, EDP Sciences (1998), Vol. 5, 145–158.

[22] D. Matignon, Can positive pseudo-differential operators of diffusive type help stabilize unstable systems? In: *Proc. of 15th International*

Symposium on Mathematical Theory of Networks and Systems, University of Notre Dame, USA, https://www3.nd.edu/∼mtns/cdrom.pdf (2002).

[23] D. Matignon, Diffusive representations for fractional Laplacian: Systems theory framework and numerical issues. *Physica Scripta* **T136** (2009), # 014009; doi: 10.1088/0031-8949/2009/T136/014009.

[24] D. Matignon, Optimal control of fractional systems: a diffusive formulation. In: *Proc. of 19th International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary (2010).

[25] G. Montseny, Diffusive representation of pseudodifferential time operators. In: *Proc.of ESAIM Fractional Differential Systems: Models, Methods and Applications*, EDP Sciences (1998), Vol. 5, 159–175.

[26] G. Montseny, J. Audounet, B. Mbodje, Optimal models of fractional integrators and applications to systems with fading memory. In: *Int. Conf. IEEE Systems, Man and Cybernetics*, Le Touquet, France, IEEE (1993).

[27] I. Petráš, Tuning and implementation methods for fractional order controllers. *Fract. Calc. Appl. Anal.* **15**, No 2 (2012), 282–303;
doi: 10.2478/s13540-012-0021-4;
at https://www.degruyter.com/view/j/fca.

[28] S. Pilipović, D. Seleši, *Mera i integral – Fundamenti teorije verovatnoće*. Zavod za udžbenike (2012).

[29] A. Pisano, M.R. Rapaić, Z.D. Jeličić, E. Usai, Sliding mode control approaches to the robust regulation of linear multivariable fractional-order systems. *International J. of Robust and Nonlinear Control* **20**, No 18 (2010), 2045–2056; doi: 10.1002/rnc.1565.

[30] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. San Diego-Boston-etc., Academic Press (1998).

[31] M.R. Rapaić, Z.D. Jeličić, Optimal control of a class of fractional heat diffusion systems. *Nonlinear Dynamics* **62**, No 1–2 (2010), 39–51; doi: 10.1007/s11071-010-9697-3.

[32] A.I. Saichev, W.A. Woyczyński, *Distributions in the Physical and Engineering Sciences, Volume I. Distributional and Fractal Calculus, Integral Transforms and Wavelets*. Birkhäuser, Boston (1997).

[33] J.L. Schiff, *The Laplace Transform – Theory and Applications*. Springer (1999).

[34] M.S. Tavezoei, Time-response analysis of fractional-order control systems: A survey on recent results. *Fract. Calc. Appl. Anal.* **17**, No 2

(2014), 440–461; doi: 10.2478/s13540-014-0179-z; at https://www.degruyter.com/view/j/fca.

[35] J.C. Trigeassou, N. Maamri, Initial conditions and initialization of linear fractional differential equations. *Signal Processing*, **91**, Issue 3 (2011), 427–436; doi:10.1016/j.sigpro.2010.03.010.

[36] J.C. Trigeassou, N. Maamri, A. Oustaloup, The infinite state approach: Origin and necessity. *Computers and Mathematics with Applications* **66**, Issue 5 (2013), 892–907; doi: 10.1016/j.camwa.2012.11.020.

[37] J.C. Trigeassou, N. Maamri, J. Sabatier, A. Oustaloup, A Lyapunov approach to stability of fractional differential equations. *Signal Processing* **91**, Issue 3 (2011), 437–445; doi: 10.1016/j.sigpro.2010.04.024.

[38] J.C. Trigeassou, N. Maamri, J. Sabatier, A. Oustaloup, State variables and transients of fractional order differential systems. *Computers and Mathematics with Applications* **64**, Issue 10 (2012), 3117–3140; doi: 10.1016/j.camwa.2012.03.099.

[39] B.J. West, M. Bologna, P. Grigolini, *Physics of Fractal Operators*. Springer (2003).

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Received: January 25, 2018

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Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **21**, No 5 (2018), pp. 1396–1419,

DOI: 10.1515/fca-2018-0073; at https://www.degruyter.com/view/j/fca.